Shooting Method for Two-Point Boundary Value Problems

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SHOOTING METHOD FOR TWO-POINT BOUNDARY VALUE PROBLEMS

by

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ABSTRACT

Shooting Method for Two-point Boundary Value Problems

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The purpose of this paper is to develop the shooting method as a technique for approximating the solution to the two-point boundary value problem on the interval $[a,b]$ with the even order differential equation (i.e. n is even)

$$u^{(n)}(t) + f(t,u(t),u^{(1)}(t,),\ldots,u^{(n-1)}(t)) = 0 \qquad (0.1)$$

and boundary conditions

$$u(a) = A \qquad (0.2)$$

$$u(b) = B \qquad (0.3)$$

and with at most n-2 other boundary conditions specified at either a or b. The basic proceedure will be illustrated by the following example.

Consider the two-point boundary value problem (0.1) (0.2) (0.3) with the additional boundary conditions

$$u^{(i)}(a) = m_i \qquad (0.4)$$

for $i = 1,\ldots,k-1,k+1,\ldots,n-1$. The first step is to find values m_1 and m_2 such that the solutions or "shots", $u_1(t)$ and $u_2(t)$, to (0.1) that satisfy the initial conditions

$$u(a) = A$$
$$\vdots$$
$$u^{(k-1)}(a) = m_{k-1}$$
$$u^{(k)}(a) = m_l$$
$$u^{(k+1)}(a) = m_{k+1}$$
$$\vdots$$
$$u^{(n-1)}(a) = m_{n-1}$$

with $l = 1,2$, respectively, with the property that

$$u_1(b) < B < u_2(b).$$

The interval $[m_1,m_2]$ is then searched by seccussive bisection to find the value, m, such that the solution or "shot", u(t), to the initial value problem with (0.1) and initial conditions

$$u(a) = A$$
$$\vdots$$
$$u^{(k-1)}(a) = m_{k-1}$$
$$u^{(k)}(a) = m_i$$
$$u^{(k+1)}(a) = m_{k+1}$$
$$\vdots$$
$$u^{(n-1)}(a) = m_{n-1}$$

has the property that $u(b) = B$. (61 pages)

CHAPTER I

INTRODUCTION

1.1 Kinds of Boundary Value Problems to be Treated

The main purpose of this paper is to develop the "shooting method" as an iterative method for approximating the solutions to two-point boundary value problems of the form

$$u^{(n)}(t) + f(t,u(t),u^{(1)}(t),\ldots,u^{(n-1)}(t)) = 0 \tag{1.1}$$

$$u(a) = A \tag{1.2}$$

$$u(b) = B \tag{1.3}$$

where n is a positive even integer. It shall be assumed that the function $f(t,u,u^{(1)},\ldots,u^{(n-1)})$ is continuous in $(t,u,u^{(1)},\ldots,u^{(n-1)})$, at least in the interior of its domain. It shall also be assumed that $f(t,u,u^{(1)},\ldots,u^{(n-1)})$ is Lipschitizian, or, that there exist n nonnegative constants K_0, $K_1,\ldots,K_{n-1}$ such that for any two points $(t,u,u^{(1)},\ldots,u^{(n-1)})$ and $(t,v,v^{(1)},\ldots,v^{(n-1)})$ in the domain of f, the inequality

$$\left|f(t,u,u^{(1)},\ldots,u^{(n-1)}) - f(t,v,v^{(1)},\ldots,v^{(n-1)})\right|$$

$$= K_0 \left|u - v\right| + \sum_{i=1}^{n-1} K_i \left|u^{(i)} - v^{(i)}\right| \tag{1.4}$$

holds.

A solution to (1.1) on the interval [a,b] is any function, u(t), defined on [a,b] that has n continuous derivatives and that satisfies

$$f(t,u(t),u^{(1)}(t),\ldots,u^{(n-1)}(t)) = - u^{(n)}(t)$$

for every $t \in [a,b]$. The first restriction is necessary since the existence of the first derivative, $u^{(1)}(t)$, implies that the function, $u(t)$, is continuous, the existence of the i-th derivative, $u^{(i)}(t)$, implies that the $(i - 1)$-th derivative, $u^{(i-1)}(t)$, is continuous, and the n-th derivative, $u^{(n)}(t)$, is continuous since f is. The second restriction is obviously necessary.

A solution of the inital value problem for (1.1) on $[a,b]$ is defined to be a solution to (1.1) that satisfies

$$
\begin{aligned}
u(a) &= A \\
u^{(1)}(a) &= u_0^{(1)} \\
u^{(2)}(a) &= u_0^{(2)} \\
&\vdots \\
u^{(n-1)}(a) &= u_0^{(n-1)}
\end{aligned}
$$

if the initial conditions are specified at $t = a$ or that satisfies

$$
\begin{aligned}
u(b) &= B \\
u^{(1)}(b) &= u_0^{(1)} \\
u^{(2)}(b) &= u_0^{(2)} \\
&\vdots \\
u^{(n-1)}(b) &= u_0^{(n-1)}
\end{aligned}
$$

if the initial conditions are specified at $t = b$.

1.2 Standard Results for Initial Value Problems

The n-th order differential equation (1.1) can be viewed as the system of differential equations

$$u_n^{(1)}(t) + f(t,u_1(t),u_2(t),\ldots,u_n(t)) = 0 \tag{1.5}$$

$$u_{n-1}^{(1)}(t) = u_n(t)$$

$$\vdots$$

$$u_1^{(1)}(t) = u_2(t)$$

where $u_1(t) = u(t)$ and $u_i(t) = u^{(i-1)}(t)$ for $i = 2,\ldots,n$. By defining

$$U(t) = (u_1(t),u_2(t),\ldots,u_n(t))$$

and

$$F(t,U(t)) = (u_2(t),u_3(t),\ldots,u_n(t),-f(t,u_1(t),\ldots,u_n(t)))$$

this system can be denoted by

$$U^{(1)}(t) = F(t,U(t)) \; . \tag{1.6}$$

Defining

$$U_0 = (u_0,u_0^{(1)},\ldots,u_0^{(n-1)})$$

the initial conditions can be denoted by

$$U(a) = U_0$$

or

$$U(b) = U_0.$$

Lemma 1.1

$F(t,U)$ is a continuous function in (t,U) if and only if $f(t,u_1,u_2,\ldots,u_n)$ is a continuous function in $(t,u_1,u_2,\ldots,u_n)$.

In each of the following standard results from the theory of ordinary differential equations that are to be referred to from time to time, it will be assumed that a domain is an open connected set. It will also be assumed that the L_1 vector norm (i.e. $\|U\| = |u_1| + |u_2| + \ldots + |u_n|$) will be used.

Lemma 1.2

If the function $F(t,U)$ is a continuous function in its domain, the initial value problem

$$U^{(1)}(t) = F(t,U(t)) \tag{1.6}$$

$$U(t_0) = U_0 \tag{1.7}$$

is equivalent to the integral equation

$$U(t) = U_0 + \int_{t_0}^{t} F(s,U(s))\, ds. \tag{1.8}$$

Proof. Integration of (1.6) immediately yields (1.8). Therefore, solutions to (1.6) (1.7) are solutions to (1.8). Converesly, let $U(t)$ be a solution to (1.8). Setting $t = t_0$ in (1.8), yields $U(t_0) = U_0$. Moreover $U(t)$ is continuous, since any integral is a continuous function of its upper limit and differention of (1.8) yields (1.6).

Theorem 1.1 (Existence of solutions)

Let $F(t,U)$ be continuous in its domain, D, which contains the point (t_0, U_0). Then there exists a function, $U(t)$, which has one continuous derivative defined on some interval I containing t_0 satisfying

$$U^{(1)}(t) = F(t, U(t)) \tag{1.6}$$

$$U(t_0) = U_0 \tag{1.7}$$

and $(t, U(t)) \in D$ for every $t \in I$.

Proof. Since D is an open set, there exist an $r > 0$ such that all points whose distance from (t_0, U_0) is less that r are contained in D. Let the region R given by

$$R = \{(t,U) \mid a \leq t \leq b \text{ and } A \leq U \leq B\}$$

be any closed rectangle containing (t_0, U_0) that is contained in this open disk of radius r.

Since F is uniformly continuous on R, for every $\varepsilon > 0$ there exist $\delta > 0$ such that if

$$\|U - V\| \leq \delta \text{ and } |t - p| \leq \delta$$

then

$$\|F(t,U) - F(p,V)\| \leq \varepsilon.$$

Divide $[t_0, b]$ into n parts such that $t_0 = r_0 < r_1 < \ldots < r_{n-1} < r_n = b$ such that

$$\max_{1 \leq k \leq n} |r_k - r_{k-1}| \leq \min(\delta, \delta/M)$$

where $M = \max_{(t,U) \in R} \|F(t,U)\|$.

Construct a piecewise linear function x_ε in the following manner. Set $x_\varepsilon(r_0) = U_0$ and the slope from r_0 to r_1 as $F(r_0,U_0)$. Let w_1 be the value of x_ε at r_1. Make the slope from r_1 to r_2, $F(r_1,w_1)$. Continue in a similar manner until r_n is reached. Now for t where $x_\varepsilon^{(1)}(t)$ exist, there exist a k such that

$$r_{k-1} \leq t \leq r_k$$

and

$$\|x_\varepsilon^{(1)}(t) - F(t,x_\varepsilon(t))\| = \|F(r_{k-1},x_\varepsilon(r_{k-1})) - F(t,x_\varepsilon(t))\| < \varepsilon$$

since $|t - r_{k-1}| \leq |r_k - r_{k-1}| \leq \delta$ and

$$\begin{aligned} \|x_\varepsilon(t) - x_\varepsilon(r_{k-1})\| &= \|F(r_{k-1},x_\varepsilon(r_{k-1}))\,(t - r_{k-1})\| \\ &\leq \|F(r_{k-1},x_\varepsilon(r_{k-1}))\|\;|t - r_{k-1}| \\ &\leq M\cdot(\delta/M) = \delta . \end{aligned}$$

A similar construction can be used for defining $x_\varepsilon(t)$ on $[a,t_0]$.

Let $\{\varepsilon_n\}$, $n = 1,2,3,\ldots$ be a monotone decreasing sequence of positive real numbers tending to zero as n approaches infinity. Let $\{x_n(t)\}$ be the corresponding sequence of piecewise continuous differentiable functions, whose existence was just proven, such that

$$\|x_n^{(1)}(t) - F(t,x_n(t))\| \leq \varepsilon_n$$

for every t where $x_n^{(1)}(t)$ is defined.

Now there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$, converging uniformly on $[a,b]$ to a limit function x(t), which is continuous since each $x_n(t)$ is continuous..

In order to prove the preceding, let $\{p_k\}$ be an enumeration of the collection of points where any $x_n^{(1)}(t)$ does not exist, which is countable since it is a countable union of finite sets. The set of numbers $\{x_n(p_1)\}$ is bounded and hence there exist a subsequence of distinct functions $\{x_{n_1}\}$ such that the subsequence $\{x_{n_1}(p_1)\}$ is convergent. Similarly, the set of numbers $\{x_{n_1}(p_2)\}$ is bounded and hence there exist a subsequence of distinct functions $\{x_{n_2}\}$ such that the subsequence $\{x_{n_2}(p_2)\}$ is convergent. Continuing in this manner, an infinite set of distinct functions $\{x_{n_k}\}$ is obtained which has the property that the sequence $\{x_{n_k}(t)\}$ converges at $t = p_1, p_2, \ldots, p_k$. Defining y_n to be the function x_{n_n}, then $\{y_n\}$ is the required sequence which is uniformily convergent on $[a,b]$.

This limit function, x(t), is a solution of (1.6) that satisfies (1.7). To see this, consider

$$x_{n_k}(t) = U_0 + \int_{t_0}^{t} (F(s,x_{n_k}(s)) + \Delta_{n_k}(s))\, ds$$

where $\Delta_{n_k}(t) = (x_{n_k}^{(1)}(t) - F(t,x_{n_k}(t)))$ for those points where $x_{n_k}^{(1)}(t)$ exist, and zero otherwise. Now

$$\|\Delta_{n_k}(t)\| < \varepsilon_{n_k} .$$

Since F is uniformly continuous on R, and the sequence $\{x_{n_k}(t)\}$ converges to x(t) uniformily on $[a,b]$ as k approaches infinity, it follows that x(t) is a solution to (1.8).

By lemma 1.2 $x(t)$ is shown to be a solution to (1.6) (1.7). This concludes the proof.

By adding the assumption, $F(t,U)$ is Lipschitizian, to the hypothesis of theorem 1.1, solutions of (1.6) (1.7) are guaranteed to be unique. First, the following convenient lemmas will be proven.

Lemma 1.2

The function $f(t,u_1,u_2,\ldots,u_n)$ satisfies the Lipschitz condition (1.4) with constants K_0, $K_1,\ldots$, K_{n-1} if and only if $F(t,U)$ satisfies the Lipschitz condition

$$\|F(t,U) - F(t,V)\| = K \|U - V\| \tag{1.9}$$

for every (t,U) and (t,V) in the domain of F and where K is defined by

$$K = \max \{K_0,\ K_1 + 1,\ K_2 + 1,\ldots,\ K_{n-1} + 1\} .$$

Proof. Assume that $f(t,u_1,u_2,\ldots,u_n)$ satisfies (1.4) with constants $K_0,K_1,\ldots,K_{n-1}$. By definition of the norm

$$\|F(t,U) - F(t,V)\| = |u_2 - v_2| + |u_3 - v_3| + \ldots +$$
$$|u_n - v_n| + |f(t,u_1,u_2,\ldots,u_n) - f(t,v_1,v_2,\ldots,v_n)|$$

Since $f(t,u_1,\ldots,u_n)$ satisfies (1.4)

$$\|F(t,U) - F(t,V)\| = K_0 |u_1 - v_1| +$$
$$(K_1 + 1)|u_2 - v_2| + \ldots + (K_{n-1} + 1)|u_n - v_n| .$$

By the definition of K

$$\|F(t,U) - F(t,V)\| \leq K (|u_1 - v_1| + \ldots + |u_n - v_n|)$$

$$= K \|U - V\|.$$

Now assume that F(t,U) satisfies (1.9). By definition of the norm, (1.9) is equivalent to

$$|u_2 - v_2| + |u_3 - v_3| + \ldots + |u_n - v_n| +$$

$$|f(t,u_1,\ldots,u_n) - f(t,v_1,\ldots,v_n)|$$

$$\leq K (|u_1 - v_1| + \ldots + |u_n - v_n|).$$

By setting $K_0 = K$ and $K_i = K - 1$ for $i = 1,2,\ldots,n-1$, one finds that $f(t,u_1,\ldots,u_n)$ satisfies the Lipschitz condition (1.4) with the constants $K_0, K_1,\ldots, K_{n-1}$. This concludes the proof.

<u>Lemma 1.4 (Gronwall Inequality)</u>

Let K be a nonnegative constant and let F and G be continuous nonnegative functions on some interval $a \leq t \leq b$ satisfying the inequality

$$F(t) \leq K + \int_a^t F(s) \cdot G(s)\, ds$$

for $a \leq t \leq b$. Then

$$F(t) \leq K \exp(\int_a^t G(s)\, ds)$$

for $a \leq t \leq b$.

Proof. Let $U(t) = K + \int_a^t F(s) \cdot G(s)\, ds$, and observe that $U(a) = K$. Then $F(t) \leq U(t)$ by hypothesis, and, by the fundamental theorem of integral calculus and because $G(t) \geq 0$

$$U^{(1)}(t) = F(t) \cdot G(t) \leq U(t) \cdot G(t)$$

for all $a \leq t \leq b$. Multiplying this inequality by

$$\exp\left(-\int_a^t G(s)\, ds\right)$$

and applying the identity

$$U^{(1)}(t) \exp\left(-\int_a^t G(s) ds\right) - U(t) \cdot G(t) \exp\left(-\int_a^t G(s) ds\right)$$
$$= \frac{d}{dt}\left(U(t) \exp\left(-\int_a^t G(s) ds\right)\right)$$

results in

$$\frac{d}{dt}\left(U(t) \exp\left(-\int_a^t G(s) ds\right)\right) \leq 0$$

Integration from a to t gives

$$0 \geq U(t) \exp\left(-\int_a^t G(s)\, ds\right) - U(a)$$

or, since $F(t) \leq U(t)$ and $U(a) = K$,

$$F(t) \leq U(t) \leq K \exp\left(\int_a^t G(s)\, ds\right)$$

for all $a \leq t \leq b$ which is the desired inequality.

Theorem 1.2 (Uniqueness of solutions)

Let F be continuous and satisfy the Lipschitz condition

$$\| F(t,U) - F(t,V) \| = K \| U - V \| \tag{1.9}$$

in its domain, D. For any point (t_0,U_0) contained in D, there exist an interval I containing t_0 and exactly one solution, U(t), defined on I of

$$U^{(1)}(t) = F(t,U(t)) \tag{1.6}$$

$$U(t_0) = U_0 \tag{1.7}$$

Proof. Let U(t) and V(t) be solutions to (1.6) (1.7) defined on the interval I and therefore solutions to the integral equations

$$U(t) = U_0 + \int_{t_0}^{t} F(s,U(s))ds$$

and

$$V(t) = U_0 + \int_{t_0}^{t} F(s,V(s))ds.$$

Subtracting the latter from the former yields

$$U(t) - V(t) = \int_{t_0}^{t} (F(s,U(s)) - F(s,V(s)))ds.$$

Taking the norm of both sides yields

$$\| U(t) - V(t) \| = \left| \int_{t_0}^{t} \| F(s,U(s)) - F(s,V(s)) \| \, ds \right| .$$

Since F satisfies the Lipschitz condition (1.9),

$$\|U(t) - V(t)\| \leq K \int_{t_0}^{t} \|U(s) - V(s)\| \, ds. \qquad (1.10)$$

Using lemma 1.4 on (1.10) with $t \geq t_0$ yields

$$\|U(t) - V(t)\| \leq 0 \cdot \exp\left(\int_{t_0}^{t} K \, ds\right) = 0 .$$

Using lemma 1.4 on (1.10) with $t_0 \geq t$ yields

$$\|U(t) - V(t)\| \leq 0 \cdot \exp\left(\int_{t}^{t_0} K \, ds\right) = 0 .$$

Therefore $U(t) \equiv V(t)$ for all $a \leq t \leq b$ or that (1.6) has an unique solution satisfying (1.7).

Theorem 1.3

Let D be a domain containing (t_0, U_0). Suppose that $F(t,U)$ is a continuous function which satisfies the Lipschitz condition (1.9) on D. Let I be a closed interval containing t_0 on which there exists an unique solution, $U(t:t_0, U_0)$, of

$$U^{(1)}(t) = F(t, U(t)) \qquad (1.6)$$

$$U(t_0) = U_0 \qquad (1.7)$$

then there exists a $\delta > 0$ such that for any point (t_1, U_1) in D satisfying

$$|t_1 - t_0| + \|U_1 - U_0\| < \delta$$

there exists a solution $U(t:t_1, U_1)$ defined on I to (1.6) satisfying $U(t_1) = U_1$. Moreover as (t_1, U_1) approaches (t_0, U_0)

$$U(t:t_1,U_1) \to U(t:t_0,U_0)$$

uniformly on I.

Proof. Let I be $[a,b]$. Choose $\varepsilon > 0$ so that the region R_1 given by

$$R_1 = \{(t,U) \mid t \in I \text{ and } |t - t_0| + \|U - U(t:t_0,U_0)\| \leq \varepsilon\}$$

is contained in D. Let $\delta > 0$ be chosen so that

$$\delta < \varepsilon \exp(-K\,(b - a))$$

where K is the Lipschitz constant in (1.9). Let the region R be given by

$$R = \{(t,U) \mid t \in I \text{ and } |t - t_0| + \|U - U(t:t_0,U_0)\| \leq \delta\}.$$

If (t_1,U_1) is contained in R, there exist a solution, $U(t:t_1,U_1)$, through (t_1,U_1) defined on an interval $(p,q) \subset I$ and this solution satisfies

$$U(t:t_1,U_1) = U_1 + \int_{t_1}^{t} F(s,U(s:t_1,U_1))\,ds$$

for $p \leq t \leq q$. Let $U_0^1 = U(t_1:t_0,U_0)$ then for $t \in I$

$$U(t:t_0,U_0) = U_0^1 + \int_{t_1}^{t} F(s,U(s:t_0,U_0))\,ds.$$

Subtracting the latter from the former yields

$$U(t:t_1,U_1) - U(t:t_0,U_0) = U_1 - U_0^1$$
$$+ \int_{t_1}^{t} (F(s,U(s:t_1,U_1)) - F(s,U(s:t_0,U_0)))\,ds$$

for $p \leq t \leq q$. Taking norms yields

$$\|U(t:t_1,U_1) - U(t:t_0,U_0)\| \leq \|U_1 - U_0^1\|$$
$$+ \int_{t_1}^{t} \|F(s,U(s:t_1,U_1)) - F(s,U(s:t_0,U_0))\| \, ds$$

Since F satisfies (1.9)

$$\|U(t:t_1,U_1) - U(t:t_0,U_0)\| \leq \|U_1 - U_0^1\|$$
$$+ K \int_{t_1}^{t} \|U(s:t_1,U_1) - U(s:t_0,U_0)\| \, ds.$$

By lemma 1.4

$$\|U(t:t_1,U_1) - U(t:t_0,U_0)\| \leq \|U_1 - U_0^1\| \exp(\int_{t_1}^{t} K \, ds)$$
$$\leq \delta \exp(K\,(b - a)) \qquad (1.11)$$

on the interval (p,q). This proves that $U(t:t_1,U_1)$ cannot leave R_1 and therefore, it can be continued to the whole interval I.

To prove this, let $U(t:t_1,U_1)$ is bound by some constant $M < \infty$ on R_1. Since $U(t:t_0,U_0)$ is uniformly continuous on R and (1.11), both limits

$$U(p^+:t_1,U_1) = \lim_{t \to p^+} U(t:t_1,U_1)$$

and

$$U(q^-:t_1,U_1) = \lim_{t \to q^-} U(t:t_1,U_1)$$

exist. This follows at once from

$$U(t:t_1,U_1) = U_1 + \int_{t_1}^{t} F(s,U(s:t_1,U_1))\,ds$$

and therefore, if $p < r_1 < r_2 < q$,

$$\| U(r_1:t_1,U_1) - U(r_2:t_1,U_1)\| = \int_{r_1}^{r_2} \|F(s,U(s:t_1,U_1))\|\,ds$$

$$\leq M\,|r_2 - r_1| \ .$$

Thus as r_1 and r_2 tend to p^+

$$U(r_1:t_1,U_1) - U(r_2:t_1,U_1) \to 0,$$

which implies by the Cauchy criterion for convergence that $U(p^+:t_1,U_1)$ exist. Similarly for $U(q^-:t_1,U_1)$.

Now (1.11) implies that $U(p^+:t_1,U_1)$ and $U(q^-:t_1,U_1)$ are in R_1 so by defining

$$\bar{U}(t) = \begin{cases} U(q^-:t_1,U_1) & t = q \\ U(t:t_1,U_1) & p < t \leq q \end{cases}$$

$\bar{U}(t)$ is a solution to (1.6) satisfying $\bar{U}(t_1) = U_1$ on $(p,q]$.

By theorem 1.1 there exist a solution $w(t)$ to (1.6) passing through $(q^-,\bar{U}(q^-))$ on $[q,q + q_0)$, $q_0 > 0$. If $U(t)$ is defined by

$$U(t) = \begin{cases} \bar{U}(t) & p < t \leq q \\ w(t) & q \leq t < q + q_0 \end{cases}$$

then $U(t)$ is a solution to (1.6) passing through (t_1, U_1) on $(p, q + q_0)$. This is obvious for $(p,q]$. For $q < t < q + q_0$ it follows from the definition of U that

$$U(t) = U(q^-) + \int_q^t F(s, U(s))\, ds.$$

But

$$U(q^-) = U_1 + \int_{t_1}^q F(s, U(s))\, ds$$

which proves the claim for $t > q$. Similarly for $t < p$.

Therefore, given $\varepsilon > 0$ and $\delta > 0$ such that

$$\delta < \varepsilon \exp(K (b - a))$$

where K is the Lipschitz constant of (1.9), if

$$\| U_1 - U_0^1 \| < \delta$$

then for every $a \leq t \leq b$,

$$\begin{aligned} \| U(t:t_1, U_1) - U(t:t_0, U_0) \| &\leq \| U_1 - U_0^1 \| \\ &\quad + \int_{t_1}^t \| F(s, U(s:t_1, U_1)) - F(s, U(s:t_0, U_0)) \| \, ds \\ &\leq \| U_1 - U_0^1 \| + K \int_{t_1}^t \| U(s:t_1, U_1) - U(s:t_0, U_0) \| \, ds. \end{aligned}$$

By lemma 1.4

$$\begin{aligned} \| U(t:t_1, U_1) - U(t:t_0, U_0) \| &\leq \| U_1 - U_0^1 \| \exp\left(\left| \int_{t_1}^t K \, ds \right| \right) \\ &\leq \delta \exp(K (b - a)) \end{aligned}$$

Hence,

$$\| U(t:t_1,U_1) - U(t:t_0,U_0) \| < \varepsilon$$

This proves convergence and the theorem.

CHAPTER II

SECOND ORDER SHOOTING METHOD

2.1 Introduction

In this chapter it will be assumed that the function $f(t,u,u^{(1)})$ is continuous on the region

$$[a,b] \times (-\infty,\infty) \times (-\infty,\infty)$$

and that the differential equation

$$u^{(2)}(t) + f(t,u(t),u^{(1)}(t)) = 0 \qquad (2.1)$$

has the properties

(1) all initial value problems have unique solutions that exist over the interval $[a,b]$, and,

(2) that no two solutions to the differential equation (2.1) shall agree in value at more than one point in the interval $[a,b]$, that is, the solutions to the two-point boundary value problem with (2.1) are unique, if they exist.

First a comparison theorem will be proven, then a uniqueness theorem for the two-point boundary value problem with (2.1) will be proven. Finally the shooting method will be developed for the two-point boundary problem with (2.1) and an example will be given.

2.2 Theorems of Comparison and Existence

Theorem 2.1 (Comparison)

Let v(t) be a twice continuously differentiable function on $[a,b]$ satisfying

$$v^{(2)}(t) + f(t,v(t),v^{(1)}(t)) > 0. \qquad (2.2)$$

(1) If u(t) is a solution to (2.1) which agrees with v(t) in both value and slope at some point $t_0 \in [a,b]$, then

$$v(t) > u(t) \qquad t \neq t_0. \qquad (2.3)$$

(2) If u(t) is a solution to (2.1) which agrees with v(t) in value at a and b, then

$$v(t) < u(t) \qquad t \neq a,b. \qquad (2.4)$$

The inequality signs may be reversed throughout.

Lemma 2.1 contains the basic information of the above theorem. It says, roughly, that statement (1) of the above theorem is true for t sufficiently near t_0.

Lemma 2.1

Let v(t) be a twice continuously differentiable function on $[a,b]$ which satisfies (2.2). For $(t,s) \in [a,b] \times [a,b]$, let u(t,s) denoth the solution of (2.1) which agrees with v(t) in value and slope at t = s. Then for every $s_0 \in [a,b]$ there exist a $\delta_0 > 0$ such that whenever $(t,s) \in [a,b] \times [a,b]$ with

$$|t - s_0| < \delta_0 \quad \text{and} \quad |s - s_0| < \delta_0$$

that

$$u^{(2)}(t,s) < v^{(2)}(t) \qquad (2.5)$$

$$u^{(1)}(t,s) > v^{(1)}(t) \qquad t < s \qquad (2.6)$$

$$u^{(1)}(t,s) < v^{(1)}(t) \qquad t > s \qquad (2.7)$$

$$u(t,s) < v(t) \qquad t \neq s \qquad (2.8)$$

Proof. The function $v(t) - u(t,s_0)$ has a local minimum at $t = s_0$ since

$$v^{(1)}(s_0) - u^{(1)}(s_0,s_0) = 0$$

and

$$\begin{aligned} v^{(2)}(s_0) - u^{(2)}(s_0,s_0) &= v^{(2)}(s_0) + f(s_0,u(s_0,s_0),u^{(1)}(s_0,s_0)) \\ &= v^{(2)}(s_0) + f(s_0,v(s_0),v^{(1)}(s_0)) \\ &> 0 \end{aligned}$$

Set $v^{(2)}(s_0) - u^{(2)}(s_0,s_0) = d_0 > 0$. Now since $v^{(2)}(t) - u^{(2)}(t,s_0)$ is a continuous function in t, there exist a $\delta_1 > 0$ such that

$$v^{(2)}(t) - u^{(2)}(t,s_0) \geq \tfrac{1}{2} d_0$$

whenever $|t - s_0| < \delta_1$. Since the solutions of (2.1) depend continuously upon its initial conditions in the sense of theorem 1.3, the function, $u(t,s)$, is continuous in (t,s). Similarly, the solution's first derivative, $u^{(1)}(t,s)$, is continuous in (t,s). Hence, so is $u^{(2)}(t,s)$. Consequently, there exist δ_0, $0 < \delta_0 < \delta_1$, such that

$$|t - s_0| < \delta_0$$

and

$$|s - s_0| < \delta_0$$

implies

$$|u^{(2)}(t,s) - u^{(2)}(t,s_0)| < \tfrac{1}{4} d_0.$$

Therefore, whenever

$$|t - s_0| < \delta_0$$

and

$$|s - s_0| < \delta_0,$$

$$v^{(2)}(t) - u^{(2)}(t,s) = \left[v^{(2)}(t) - u^{(2)}(t,s_0)\right] + \left[u^{(2)}(t,s_0) - u^{(2)}(t,s)\right]$$

$$= \tfrac{1}{2} d_0 - \tfrac{1}{4} d_0 > 0.$$

This proves (2.5). (2.6) is derived from (2.5) by integrating from t to s. (2.7) is derived by integrating (2.5) from s to t. (2.8) is derived from (2.6) and (2.7) by integrating from t to s and s to t, respectively.

Proof of theorem 2.1. First (1) shall be proven. By lemma 2.1, every point $s_0 \in [a,b]$ is contained in some open interval of length δ_0 with the property that

$$u(t,s) < v(t) \qquad t \neq s$$

whenever t and s are both in the same interval. Since $[a,b]$ is compact, it can be covered by a finite number of such intervals. Therefore, it is sufficient to prove that the union of two overlapping intervals having property (2.8) also has this property. To avoid unnecessary complications in notation it shall be assumed that a and b are the left and right endpoints, respectively, of the two overlapping intervals.

Let $a < c < b$ where c is contained in the intersection of the overlapping intervals. Then any t in the union and c are both contained in either one interval or the other so

$$u(t,c) < v(t) \qquad t \neq c. \tag{2.9}$$

Let s be contained in $[a,c)$, then

$$u(t,s) < v(t) \qquad t \neq s \tag{2.10}$$

for all $t \in [a,c]$. In particular,

$$u(c,s) < v(c). \tag{2.11}$$

Hence, there exist some point d, $s < d < c$ at which

$$u(d,s) = u(d,c) \tag{2.12}$$

Because of the assumption that no two solutions of (2.1) can meet more than once in $[a,b]$ it follows that for t, $d < t \leq b$, that

$$u(t,s) < u(t,c).$$

This together with (2.8) and (2.9) gives

$$u(t,s) < v(t) \qquad\qquad t \neq s.$$

Obviously, the same sort of proof works for $s \in (c,b]$. Thus (1) is true.

Proof of (2): Suppose on the contrary that at some point $c \in (a,b)$

$$u(c) < v(c) \,. \tag{2.13}$$

Defining $u(t,s)$ as in lemma 2.1, by (1)

$$u(t,c) < v(t) \qquad t \neq c \tag{2.14}$$

In particular,

$$u(a,c) < v(a) = u(a)$$

and

$$u(b,c) < v(b) = u(b).$$

Hence, there exist points t_1, t_2 with $a < t_1 < t_2 < b$ such that

$$u(t_1,c) = u(t_1)$$

and

$$u(t_2,c) = u(t_2)$$

which contradicts the assumption that no two solutions of (2.1) can meet more than once in $[a,b]$. This implies that

$$v(t) \leq u(t) \qquad t \in (a,b). \tag{2.15}$$

Finally, to see that equality cannot hold in (2.15) at any point of (a,b) note the following. Equality in (2.15) requires equality in the derivative as well since this point would be a local minimun of the function $u(t) - v(t)$. This also implies that $u^{(1)}(t) - v^{(1)}(t) = 0$ at the above point, but this possiblity is ruled out by (1). This proves (2).

Theorem 2.2 (Existence and boundedness)

Let $u_1(t)$ and $u_2(t)$ be solutions to

$$u^{(2)}(t) + f(t,u(t),u^{(1)}(t)) = 0 \tag{2.1}$$

$$u(a) = A \tag{2.16}$$

on $[a,b]$. Assume that $u_1(t) < u_2(t)$ for $t \in (a,b]$.

(1) For every $B \in (u_1(b),u_2(b))$ there exist a solution, $u(t)$, to (2.1) (2.16) such that

a) $u(b) = B$

b) $u_1^{(1)}(a) < u^{(1)}(a) < u_2^{(1)}(a)$.

(2) Every solution, $u(t)$, of (2.1) (2.16) such that

$$u_1^{(1)}(a) < u^{(1)}(a) < u_2^{(1)}(a)$$

has the property that

$$u_1(t) < u(t) < u_2(t)$$

for $t \in (a,b]$.

Proof. First (1) shall be proven. By theorem 1.3, the solution of the initial value problems with (2.1) are continuously dependent upon the initial conditions. In particular, $u_1(b)$ is a continuous function in $(u_1(a),u_1^{(1)}(a))$. By the intermediate value theorem there exist a point $(m,k) \in (u_1(a),u_2(a)) \times (u_1^{(1)}(a),u_2^{(1)}(a))$ such that the solution, $u(t)$, to the initial value problem

$$u^{(2)}(t) + f(t,u(t),u^{(1)}(t)) = 0$$

$$u(a) = m$$

$$u^{(1)}(a) = k$$

has the property that $u(b) = B$. This proves (1)

Next the proof of (2). Assume that there exist a t_0, $a < t_0 \leq b$ such that

$$u_2(t_0) = u(t_0),$$

but this contradicts the assumption that no two solutions to (2.1) can agree at more than one poiht in $[a,b]$. Since solutions to (2.1) are continuous in t,

$$u_2(t) > u(t)$$

for $t \in (a,b]$. A similar proof will show that for $t \in (a,b]$

$$u_1(t) < u(t).$$

2.3 Develθping the Shooting Method

The shooting method for the two-point boundary value problem with (2.1) can be set up in the following manner. The function $f(t,u,u^{(1)})$ has been assumed to satisfy the Lipschitz property (1.4) for n = 2, i.e.

$$\left|f(t,u,u^{(1)}) - f(t,v,v^{(1)})\right| < K_0 \left|u - v\right| + K_1 \left|u^{(1)} - v^{(1)}\right| \quad (2.17)$$

holds for all $(t,u,u^{(1)})$ and $(t,v,v^{(1)})$ in the domain of f. Also since $f(t,u,u^{(1)})$ is assumed to be continuous on the region

$$[a,b] \times (-\infty,\infty) \times (-\infty,\infty) ,$$

$(t,v,v^{(1)})$ can be selected to be (t,0,0) so (2.17) becomes

$$\left|f(t,u,u^{(1)}) - f(t,0,0)\right| < K_0 \left|u\right| + K_1 \left|u^{(1)}\right| .$$

This is equivalent to the two equation

$$f(t,u,u^{(1)}) < f(t,0,0) + K_0 \left|u\right| + K_1 \left|u^{(1)}\right|$$

and

$$f(t,u,u^{(1)}) > f(t,0,0) - K_0 \left|u\right| - K_1 \left|u^{(1)}\right|.$$

Adding $u^{(2)}(t)$ to both sides of each inequality yields

$$u^{(2)}(t) + f(t,u,u^{(1)}) < u^{(2)}(t) + K_1 \left|u^{(1)}\right| + K_0 \left|u\right| + f(t,0,0)$$

and

$$u^{(2)}(t) + f(t,u,u^{(1)}) > u^{(2)}(t) - K_1 \left|u^{(1)}\right| - K_0 \left|u\right| + f(t,0,0) .$$

Let $u(t)$, $v_1(t)$, and $v_2(t)$ be twice continuously differentiable functions such that their value at $t = a$ is A and their value at $t = b$ is B be solutions to

$$u^{(2)}(t) + f(t,u(t),u^{(1)}(t)) = 0 ,$$

$$v_1^{(2)}(t) + K_1 v_1^{(1)}(t) + K_0 v_1(t) + f(t,0,0) = 0 ,$$

and

$$v_2^{(2)}(t) - K_1 v_2^{(1)}(t) - K_0 v_2(t) + f(t,0,0) = 0,$$

respectively. Now

$$v_1^{(2)}(t) + f(t,v_1(t),v_1^{(1)}(t)) > 0$$

and

$$v_2^{(2)}(t) + f(t,v_2(t),v_2^{(1)}(t)) < 0$$

so by theorem 2.1 (2)

$$v_1(t) < u(t) < v_2(t) \tag{2.18}$$

for $t \in (a,b)$. Let $u_1(t)$ and $u_2(t)$ be solutions to (2.1) that agrees with $v_1(t)$ and $v_2(t)$, respectively, in value and slope at $t = a$. By theorem 2.1 (1) and (2.18)

$$u_1(t) < u(t) < u_2(2)$$

for $t \in (a,b]$. At this point it is convenient to note that by theorem 2.2 (1) and the assumption that all initial value problems have unique solutions over the interval [a,b], the existence of $u_1(t)$ and $u_2(t)$ is sufficient for the existence of a solution, $u(t)$, to the two-point boundary value problem

$$u^{(2)}(t) + f(t,u(t),u^{(1)}(t)) = 0 \qquad (2.1)$$

$$u(a) = A \qquad (2.19)$$

$$u(b) = B \qquad (2.20)$$

The proceedure now that $u_1(t)$ and $u_2(t)$ have been found is the following iterative scheme.

(1) Set $m = [u_1(a) + u_2(a)] / 2$.

(2) Find the solution, $v(t)$, to the initial value problem

$$v^{(2)}(t) + f(t,v(t),v^{(1)}(t)) = 0 \qquad (2.1)$$

$$v(a) = A \qquad 92.20)$$

$$v^{(1)}(a) = m \; . \qquad (2.21)$$

Theorem 2.2 (2) says that

$$u_1(t) < v(t) < u_2(t) \qquad a < t \leq b \; .$$

(3) If $v(b) = B$ stop, for the desired solution has been found. If $v(b) \neq B$, then replace $u_1(t)$ by $v(t)$ if $v(b) < B$, otherwise replace $u_2(t)$ by $v(t)$ and repeat the above steps.

The above process is terminated when $v(b)$ becomes sufficiently close to B.

2.4 Example

The problem

$$y^{(2)}(t) + \left[\frac{2 + y^2(t)}{1 + y^2(t)}\right] y(t) = 1 \tag{2.22}$$

$$y(0) = 0 \tag{2.23}$$

$$y(2) = 3 \tag{2.24}$$

will be used to illustrate the basic idea of the shooting method using bisection. First, the Lipschitz constants will be found. Then the bounding curves, $v_1(t)$ and $v_2(t)$, will be found and finally, the iterative procedure of section 2.3 will be used.

Instead of using the Lipschitz condition (1.4), a more general Lipschitz condition given by

$$G_1(y - x, y^{(1)} - x^{(1)}) \leqq f(t,y,y^{(1)}) - f(t,x,x^{(1)}) \leqq G_2(y - x, y^{(1)} - x^{(1)})$$

where

$$G_1(u,u^{(1)}) = \begin{cases} L_1 u + L_2 u^{(1)} & u \geqq 0,\ u^{(1)} \geqq 0 \\ L_1 u + K_2 u^{(1)} & u \geqq 0,\ u^{(1)} \leqq 0 \\ K_1 u + L_2 u^{(1)} & u \leqq 0,\ u^{(1)} \geqq 0 \\ K_1 u + K_2 u^{(1)} & u \leqq 0,\ u^{(1)} \leqq 0 \end{cases}$$

and

$$G_2(u,u^{(1)}) = \begin{cases} K_1 u + K_2 u^{(1)} & u \geqq 0,\ u^{(1)} \geqq 0 \\ K_1 u + L_2 u^{(1)} & u \geqq 0,\ u^{(1)} \leqq 0 \\ L_1 u + K_2 u^{(1)} & u \leqq 0,\ u^{(1)} \geqq 0 \\ L_1 u + L_2 u^{(1)} & u \leqq 0,\ u^{(1)} \leqq 0 \end{cases}$$

will be used because it yields a tighter bound on the solution to (2.2). The Lipschitz constants corresponding to (2.22) are obviously $L_1 = L_2 = K_2 = 0$, $K_1 = 2$. The bounding curves, $v_1(t)$ and $v_2(t)$, are solutions of

$$v_2^{(2)}(t) + 2\, v_2(t) - 1 = 0$$

$$v_1^{(2)}(t) - 1 = 0$$

which satisfy the boundary conditions (2.23) (2.24). Actually,

$$v_2(t) = \left[\frac{2 + \cos 2\sqrt{2}}{\sin 2\sqrt{2}}\right] \sin \sqrt{2}\, t - \cos \sqrt{2}\, t + 1$$

$$v_1(t) = \tfrac{1}{2} t^2 + \tfrac{1}{2} t.$$

This implies that $u_1(t)$ is the solution to (2.22) (2.23) such that $u_1(0) = \frac{1}{2}$, and $u_2(t)$ is the solution to (2.22) (2.23) such that $u_2(0) = \sqrt{2}\,(2 + \cos 2\sqrt{2})/(\sin 2\sqrt{2})$.

The starting high shot, $u_2(t)$, and low shot, $u_1(t)$, are shown in Fig. 3.1 with the first five shots.

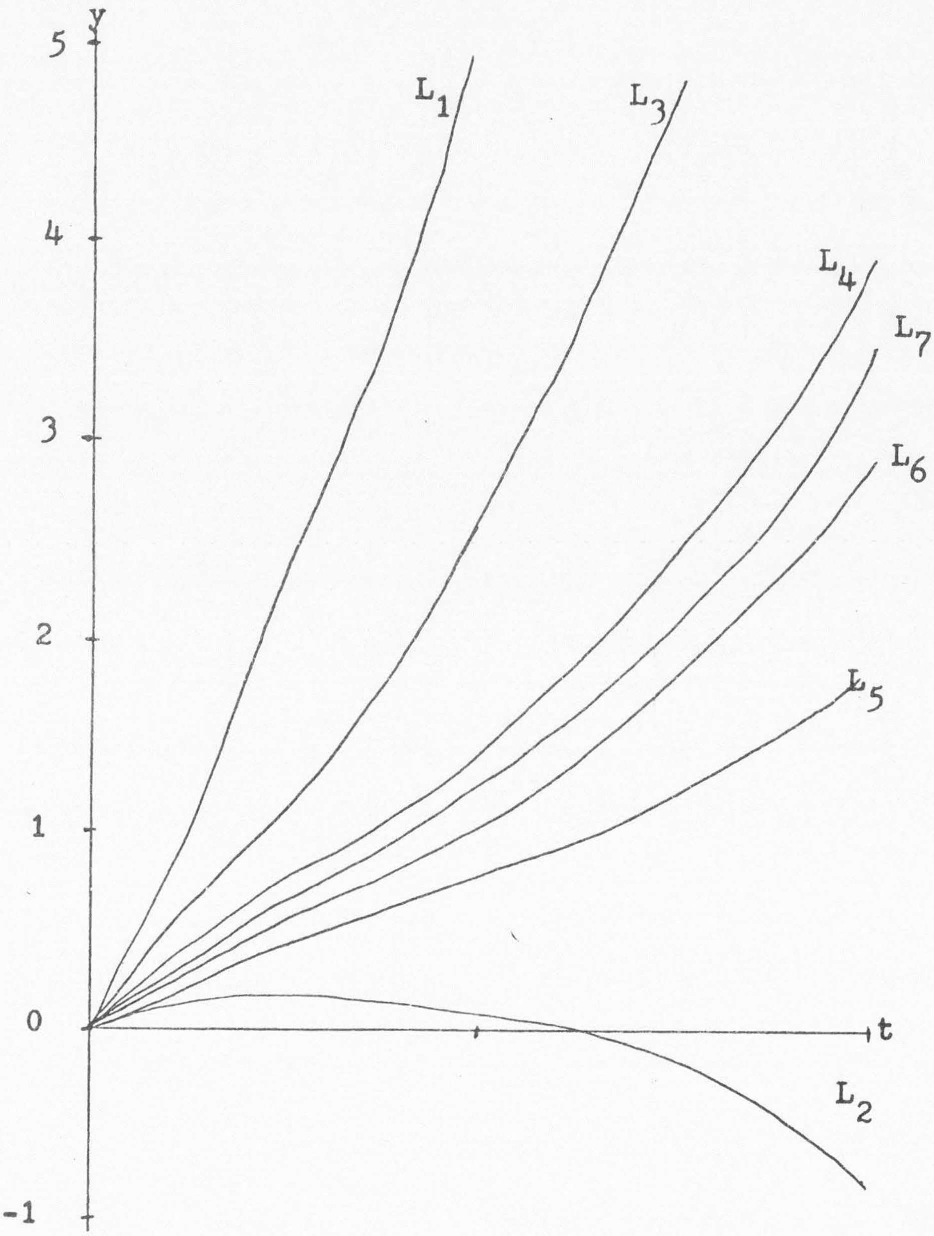

Figure 2.1. Starting high "shot" L_1, starting low "shot" L_2, and the first five "shots" L_3 - L_7.

CHAPTER III

FOURTH ORDER SHOOTING METHOD

3.1 Introduction

In this chapter it will be assumed that $f(t,u,u^{(1)},u^{(2)}, u^{(3)})$ is continuous on the region

$$[a,b] \times (-\infty,\infty) \times (-\infty,\infty) \times (-\infty,\infty) \times (-\infty,\infty).$$

Furthermore, it shall be assumed that the differential equation

$$u^{(4)}(t) + f(t,u(t),u^{(1)}(t),u^{(2)}(t),u^{(3)}(t)) = 0 \qquad (3.1)$$

has the properties that

(1) all initial value problems have unique solutions which exist over the interval $[a,b]$, and

(2) that no two solutions to the differential equation (3.1) can agree in value at more that one point in $[a,b]$, that is, the two-point boundary value problem with (3.1) has unique solutions, if they exist.

First a comparison theorem will be proven, then an uniqueness theorem for the two-point boundary value problem with (3.1) will be proven, Finally the shooting method will be developed for the two-point boundary value problem with (3.1) and two examples will be given.

3.2 Theorems of Comparison and Existence

Theorem 3.1 (Comparison of solutions)

Let $f(t,u,u^{(1)},u^{(2)},u^{(3)})$ be continuous in $(t,u,u^{(1)},u^{(2)},u^{(3)})$ and $v(t)$ be a solution to

$$v^{(4)}(t) + f(t,v(t),v^{(1)}(t),v^{(2)}(t),v^{(3)}(t)) > 0. \qquad (3.2)$$

(1) If $u(t)$ is the solution to (3.1) that agrees with $v(t)$ in value and three derivatives at some point $t_0 \in [a,b]$, then

$$v(t) > u(t) \qquad t \neq t_0. \qquad (3.3)$$

(2) If $u(t)$ is the solution to (3.1) that agrees with $v(t)$ in value at both a and b then

$$v(t) \leq u(t) \qquad t \neq a,b. \qquad (3.4)$$

The inequality signs may be reversed throughout this theorem.

Lemma 3.1 contains the basic information of the above theorem. It says, roughly, that statement (1) is true for t sufficiently near t_0.

Lemma 3.1

Let $v(t)$ be a four times continuously differentiable function on a,b which satisfies (3.2). For $(t,s) \in [a,b] \times [a,b]$, let $u(t,s)$ denote the solution of (3.1) which agrees with $v(t)$ in value and three derivatives at $t = s$. Then for every $s_0 \in [a,b]$ there exist a $\delta_0 > 0$ such that whenever $(t,s) \in [a,b] \times [a,b]$ with

$$|t - s_0| < \delta_0 \qquad \text{and} \qquad |s - s_0| < \delta_0$$

then

$$u^{(4)}(t,s) < v^{(4)}(t) \tag{3.5}$$

$$u^{(3)}(t,s) < v^{(3)}(t) \qquad t > s \tag{3.6}$$

$$u^{(3)}(t,s) > v^{(3)}(t) \qquad t < s \tag{3.7}$$

$$u^{(2)}(t,s) < v^{(2)}(t) \qquad t \neq s \tag{3.8}$$

$$u^{(1)}(t,s) < v^{(1)}(t) \qquad t > s \tag{3.9}$$

$$u^{(1)}(t,s) > v^{(1)}(t) \qquad t < s \tag{3.10}$$

$$u(t,s) < v(t) \qquad t \neq s. \tag{3.11}$$

<u>Proof.</u> The function (of t) $v(t) - u(t,s_0)$ has a local minimum at $t = s_0$ since

$$v^{(1)}(s_0) - u^{(1)}(s_0,s_0) = 0$$

$$v^{(2)}(s_0) - u^{(2)}(s_0,s_0) = 0$$

$$v^{(3)}(s_0) - u^{(3)}(s_0,s_0) = 0$$

and

$$\begin{aligned} v^{(4)}(s_0) - u^{(4)}(s_0,s_0) &= v^{(4)}(s_0) + f(t,u(s_0,s_0),\ldots,u^{(3)}(s_0,s_0)) \\ &= v^{(4)}(s_0) + f(t,v(s_0),\ldots,v^{(3)}(s_0)) > 0. \end{aligned}$$

Set $v^{(4)}(s_0) - u^{(4)}(s_0,s_0) = d_0 > 0$. Now, since $v^{(4)}(t) - u^{(4)}(t,s_0)$ is a continuous function in t, there exist a $\delta_1 > 0$ such that

$$v^{(4)}(t) - u^{(4)}(t,s_0) > \tfrac{1}{2} d_0 > 0$$

for $|t - s_0| < \delta_1$. Also, since the solutions of (3.1) depend continuously upon their initial conditions in the sense of theorem 1.3, as do the derivatives $u^{(1)}(t,s)$, $u^{(2)}(t,s)$, and $u^{(3)}(t,s)$, the functions $u(t,s)$, $u^{(1)}(t,s)$, $u^{(2)}(t,s)$, $u^{(3)}(t,s)$ are continuous in (t,s). Hence, so is $u^{(4)}(t,s)$. Consequently, there exist a δ_0, $0 < \delta_0 < \delta_1$, such that

$$|u^{(4)}(t,s) - u^{(4)}(t,s_0)| < \tfrac{1}{4} d_0$$

whenever $|t - s_0| < \delta_0$ and $|s - s_0| < \delta_0$. Therefore, whenever $|t - s_0| < \delta_0$ and $|s - s_0| < \delta_0$,

$$v^{(4)}(t) - u^{(4)}(t,s) = \left[v^{(4)}(t) - u^{(4)}(t,s_0)\right] + \left[u^{(4)}(t,s_0) - u^{(4)}(t,s)\right]$$

$$\geq \tfrac{1}{2} d_0 - \tfrac{1}{4} d > 0$$

This proves (3.5). (3.6) is derived by integrating (3.5) from t to s. (3.7) is derived from (3.5) by integrating from s to t. (3.8) is derived from (3.6) and (3.7) in a similar manner. (3.9) and (3.10) are derived from (3.8) by integration. (3.11) is derived from (3.9) and (3.10) by integration.

<u>Proof of theorem 3.1.</u> First (1) shall be proven. By lemma 3.1, every point $s_0 \in [a,b]$ is contained in some open interval δ_0 with the property that

$$u(t,s) < v(t) \qquad t \neq s \qquad (3.12)$$

whenever t and s are both in this interval. Since $[a,b]$ is compact, it can be covered with a finite number of such intervals. Therefore, it is sufficient to prove that the union of two overlapping intervals having property (3.12) also has this property. To avoid unnecessarily complicated notation it shall be assumed that a and b are the left and right endpoints, respectively, of two such overlapping intervals.

Let $a < c < b$ where c is contained in the intersection of the overlapping intervals. Then any t in the union and c are both in either one interval or the other so

$$v(t) > u(t,c) \qquad t \neq c. \tag{3.13}$$

Let s be contained in $[a,c)$, then

$$v(t) > u(t,s) \qquad t \neq s \tag{3.14}$$

for $a \leq t \leq c$. In particular,

$$v(c) > u(c,s) \; . \tag{3.15}$$

Hence, there exists some point d, $s \leq d \leq c$, at which

$$u(d,s) = u(d,c). \tag{3.16}$$

Now because of the assumption that no two solutions of (3.1) can meet more than once in $[a,b]$, it follows that for $t \in [d,b]$ that

$$u(t,s) < u(t,c) \; .$$

This together with (3.13) and (3.12) gives

$$u(t,s) < v(t) \qquad\qquad t \neq s.$$

Obviously the same sort of proof will work for $s \in (c,b]$. Thus (1) is true.

Next the proof of (2). Suppose on the contrary that at some point c contained in (a,b)

$$u(c) < v(c). \tag{3.17}$$

Defining u(t,s) as in lemma 3.1, by (1)

$$u(t,c) < v(t) \qquad\qquad t \neq c. \tag{3.18}$$

in particular,

$$u(a,c) < v(a) = u(a)$$

and

$$u(b,c) < v(b) = u(b).$$

Hence, there exist point t_1, t_2 with $a < t_1 < c$ and $c < t < b$ such that

$$u(t_1,c) = u(t_1)$$

and

$$u(t_2,c) = u(t_2)$$

which contradicts the assumption that no two solutions of (3.1) can meet more than once in $[a,b]$. This implies that

$$v(t) \leq u(t)$$

for $t \in (a,b)$. Thus the theorem is true.

Theorem 3.2 (Properties of existence and boundedness of solutions)

Let $u_1(t)$ and $u_2(t)$ be solutions to

$$u^{(4)}(t) + f(t,u(t),u^{(1)}(t),u^{(2)}(t),u^{(3)}(t)) = 0 \qquad (3.1)$$

$$u(a) = A \qquad (3.19)$$

on $[a,b]$. Assume that $u_1(t) < u_2(t)$ for $t \neq a$.

(1) For every point $B \in (u_1(b),u_2(b))$, there exist a solution, $u(t)$, to (3.1) (3.19) such that

$$\text{a) } u(b) = B \qquad (3.20)$$

$$\text{b) } (u(a),u^{(1)}(a),u^{(2)}(a),u^{(3)}(a)) = c\,(u_1(a),u_1^{(1)}(a),u_1^{(2)}(a),u_1^{(3)}(a))$$

$$+ (1 - c)\,(u_2(a),u_2^{(1)}(a),u_2^{(2)}(a),u_2^{(3)}(a)) \qquad (3.21)$$

where $c \in (0,1)$.

(2) If $u(t)$ is a solution to (3.1) (3.19) such that

$$u_1^{(3)}(a) \leq u^{(3)}(a) \leq u_2^{(3)}(a)$$

$$u_1^{(2)}(a) \leq u^{(2)}(a) \leq u_2^{(2)}(a)$$

$$u_1^{(1)}(a) \leq u^{(1)}(a) \leq u_2^{(1)}(a)$$

with strict inequality holding for at least one of the above conditions, then for $a < t \leq b$

$$u_1(t) < u(t) < u_2(t) \tag{3.22}$$

Proof. First (1) will be proven. By theorem 1.3, the solutions to the initial value problems with (3.1) are continuously dependent upon their initial conditions. In particular, $u_1(b)$ is a continuous function of $(u_1(a), u_1^{(1)}(a), u_1^{(2)}(a), u_1^{(3)}(a))$. By the intermediate value theorem, there exist a point

$$(A,m,k,l) = c\,(A, u_1^{(1)}(a), u_1^{(2)}(a), u_1^{(3)}(a)) + (1 - c)\,(A, u_2^{(1)}(a), u_2^{(2)}(a), u_2^{(3)}(a)) \tag{3.23}$$

where $c \in (0,1)$ such that the solution, $u(t)$, to the initial value problem

$$u^{(4)}(t) + f(t,u(t),u^{(1)}(t),u^{(2)}(t),u^{(3)}(t)) = 0 \tag{3.1}$$

$$u^{(3)}(a) = l \tag{3.24}$$

$$u^{(2)}(a) = k \tag{3.25}$$

$$u^{(1)}(a) = m \tag{3.26}$$

$$u(a) = A \tag{3.27}$$

is the desired solution that satisfies (3.20) and (3.21). This proves (1).

Next to prove (2), by contradiction assume that there exists a $t_0 \in (a,b]$ such that

$$u_2(t_0) = u(t_0)$$

but this contradicts the assumption that no two solutions to (3.1) can meet more than once in $[a,b]$. By the continuity of the solutions of (3.1),

$$u(t) < u_2(t)$$

for $a < t \leq b$. A similar argument will show that

$$u_1(t) < u(t)$$

for $a < t \leq b$. Therefore, (3.22) holds and the proof of this theorem is finished.

3.3 Developing the Shooting Method

The shooting method for the fourth order differential equation two-point boundary value problem can be set up in the following manner. The function $f(t,u,u^{(1)},u^{(2)},u^{(3)})$ has been assumed to satisfy the Lipschitz condition (1.4) for $n = 4$, i.e.

$$\left|f(t,u,u^{(1)},u^{(2)},u^{(3)}) - f(t,v,v^{(1)},v^{(2)},v^{(3)})\right| \leq K_0 \left|u - v\right| + K_1 \left|u^{(1)} - v^{(1)}\right| + K_2 \left|u^{(2)} - v^{(2)}\right| + K_3 \left|u^{(3)} - v^{(3)}\right| \qquad (3.28)$$

holds for all $(t,u,u^{(1)},u^{(2)},u^{(3)})$ and $(t,v,v^{(1)},v^{(2)},v^{(3)})$ in the domain of f. Now $(t,v,v^{(1)},v^{(2)},v^{(3)})$ can be set equal to $(t,0,0,0)$ since f is continuous on the region

$$[a,b] \times (-\infty,\infty) \times (-\infty,\infty) \times (-\infty,\infty) \times (-\infty,\infty)$$

so (3.28) becomes

$$\left| f(t,u,u^{(1)},u^{(2)},u^{(3)}) - f(t,0,0,0,0)\right| < K_0 \left|u\right| + K_1 \left|u^{(1)}\right| + K_2 \left|u^{(2)}\right| + K_3 \left|u^{(3)}\right| .$$

This is equivalent to the two equations

$$f(t,u,u^{(1)},u^{(2)},u^{(3)}) < f(t,0,0,0,0) + K_0 \left|u\right| + K_1 \left|u^{(1)}\right| + K_2 \left|u^{(2)}\right| + K_3 \left|u^{(3)}\right|$$

and

$$f(t,u,u^{(1)},u^{(2)},u^{(3)}) > f(t,0,0,0,0) - K_0 \left|u\right| - K_1 \left|u^{(1)}\right| - K_2 \left|u^{(2)}\right| - K_3 \left|u^{(3)}\right| .$$

Adding $u^{(4)}$ to both sides of each inequality yields

$$u^{(4)} + f(t,u,u^{(1)},u^{(2)},u^{(3)}) < u^{(4)} + K_3\,|u^{(3)}| + K_2\,|u^{(2)}|$$

$$+ K_1\,|u^{(1)}| + K_0\,|u| + f(t,0,0,0,0)$$

and

$$u^{(4)} + f(t,u,u^{(1)},u^{(2)},u^{(3)}) > u^{(4)} - K_3\,|u^{(3)}| - K_2\,|u^{(2)}|$$

$$- K_1\,|u^{(1)}| - K_0\,|u| + f(t,0,0,0,0).$$

Let $u(t)$, $v_1(t)$, $v_2(t)$ be the solutions to

$$u^{(4)}(t) + f(t,u(t),u^{(1)}(t),u^{(2)}(t),u^{(3)}(t)) = 0\ ,$$

$$v_1^{(4)}(t) + K_3\,v_1^{(3)}(t) + K_2\,v_1^{(2)}(t) + K_1\,v_1^{(1)}(t) +$$

$$+ K_0\,v_1(t) + f(t,0,0,0,0) = 0\ ,$$

and

$$v_2^{(4)}(t) - K_3\,v_2^{(3)}(t) - K_2\,v_2^{(2)}(t) - K_1\,v_2^{(1)}(t) -$$

$$- K_0\,v_2(t) + f(t,0,0,0,0) = 0\ ,$$

respectively, that satisfies the specified boundary conditions of the two-point boundary value problem. Now

$$v_1^{(4)}(t) + f(t,v_1(t),v_1^{(1)}(t),v_1^{(2)}(t),v_1^{(3)}(t)) \geqslant 0$$

and

$$v_2^{(4)}(t) + f(t,v_2(t),v_2^{(1)}(t),v_2^{(2)}(t),v_2^{(3)}(t)) < 0$$

so by theorem 3.1 (2)

$$v_1(t) \leq u(t) \leq v_2(t) \qquad (3.29)$$

for $a < t < b$.

Let $u_1(t)$ and $u_2(t)$ be solutions to (3.1) that agree with $v_1(t)$ and $v_2(t)$, respectively, in value and their first three derivatives at $t = a$. By theorem 3.1 (1) and (3.29)

$$u_1(t) < u(t) < u_2(t) \qquad a < t \leq b.$$

At this point it is convenient to note that by theorem 3.2 (1) and the assumption that all initial value problems have unique solutions over the interval $[a,b]$. the existence of $u_1(t)$ and $u_2(t)$ is sufficient for the existence of a solution, $u(t)$, to the two-point boundary value problem

$$u^{(4)}(t) + f(t,u(t),u^{(1)}(t),u^{(2)}(t),u^{(3)}(t)) = 0 \qquad (3.1)$$

$$u(a) = A \qquad (3.19)$$

$$u(b) = B. \qquad (3.20)$$

The proceedure now that $u_1(t)$ and $u_2(t)$ have been found is the following iterative scheme.

(1) Let $m = [u_1^{(1)}(a) + u_2^{(1)}(a)] / 2$,

$k = [u_1^{(2)}(a) + u_2^{(2)}(a)] / 2$,

$l = [u_1^{(3)}(a) + u_2^{(3)}(a)] / 2$.

(2) Find the solutions, $v(t)$, to the initial value problem

$$v^{(4)}(t) + f(t,v(t),v^{(1)}(t),v^{(2)}(t),v^{(3)}(t)) = 0 \qquad (3.1)$$

$$v(a) = A \tag{3.19}$$

$$v^{(1)}(a) = m \tag{3.30}$$

$$v^{(2)}(a) = k \tag{3.31}$$

$$v^{(3)}(a) = 1. \tag{3.32}$$

Theorem 3.2 (2) says that

$$u_1(t) < v(t) < u_2(t)$$

for $a < t \leq b$.

(3) If $v(b) = B$ stop, for the desired solution has been found. If $v(b) \neq B$, then replace $u_1(t)$ by $v(t)$ if $v(b) < B$, otherwise replace $u_2(t)$ by $v(t)$ and repeat the above steps. The above process is terminated when $v(b)$ becomes sufficiently close to B.

3.4 Examples

Example 1

The problem

$$y^{(4)}(t) + \left[\frac{9 + (y^{(2)}(t))^2}{1 + (y^{(2)}(t))^2}\right] y^{(2)}(t) - \left[\frac{16 + y^2(t)}{1 + y^2(t)}\right] y(t) = 144 \qquad (3.33)$$

$$y(0) = 0 \qquad (3.34)$$

$$y^{(1)}(0) = -6 \qquad (3.35)$$

$$y^{(2)}(0) = 36 \qquad (3.36)$$

$$y(2) = 3 \qquad (3.37)$$

will be used to illustrate the shooting method of section 3.3. First, the Lipschitz constants will be found. Then the bounding curves, $v_1(t)$ and $v_2(t)$, will be found and finally, the iterative proceedure will be used.

Instead of using the Lipschitz condition (1.4), a more general Lipschitz condition given by

$$d_0(y - x) + d_1(y^{(1)} - x^{(1)}) + d_2(y^{(2)} - x^{(2)}) + d_3(y^{(3)} - x^{(3)})$$

$$\leq f(t,y,y^{(1)},y^{(2)},y^{(3)}) - f(t,x,x^{(1)},x^{(2)},x^{(2)}) \leq$$

$$g_0(y - x) + g_1(y^{(1)} - x^{(1)}) + g_2(y^{(2)} - x^{(2)}) + g_3(y^{(3)} - x^{(3)})$$

where

$$d_i(u) = \begin{cases} L_i\, u & u \geq 0 \\ K_i\, u & u < 0 \end{cases}$$

and

$$g_1(u) = \begin{cases} L_1 \; u & u < 0 \\ K_1 \; u & u \geq 0 \end{cases}$$

will be used because it yields a tighter bound on the solution to (3.3). The Lipschitz constants corresponding to (3.33) are obviously $L_1 = L_2 = L_3 = K_0 = K_1 = K_3 = 0$, $L_0 = -16$ and $K_2 = 9$. The bounding curves, $v_1(t)$ and $v_2(t)$, are therefore solutions of

$$\overset{(4)}{v_2}(t) + 9 \overset{(2)}{v_2}(t) - 144 = 0$$

and

$$\overset{(4)}{v_1}(t) - 16 \, v_1(t) - 144 = 0$$

which satisfy the boundary conditions (3.34) (3.35) (3.36) (3.37). Actually,

$$v_2(t) = -10 \cos 3t \,/\, 9 + B \sin 3t + C \, e^{3t} + D \, e^{-3t} + 8 \, t^2$$

and

$$v_1(t) = R \sin 2t - 9 \cos 2t + S \, e^{2t} - S \, e^{-2t}$$

where

$$B = \left[\frac{20 \cos 6 + 8 \, e^{6} - 28 \, e^{-6} - 522}{9 \, e^{-6} + 18 \sin 6 - 9 \, e^{6}}\right]$$

$$C = \left[\frac{10 \, e^{-6} + 261 - 8 \sin 6 - 10 \cos 6}{9 \, e^{-6} + 18 \sin 6 - 9 \, e^{6}}\right]$$

$$D = \left[\frac{10 \cos 6 + 28 \sin 6 - 10 \, e^{6} - 261}{9 \, e^{-6} + 18 \sin 6 - 9 \, e^{6}}\right]$$

$$R = \left[\frac{12 + 3\,e^{-4} - 3\,e^{4} - 18\cos 4}{e^{4} - e^{-4} - 2\sin 4}\right]$$

$$S = \left[\frac{9\cos 4 + 3\sin 4 - 6}{e^{4} - e^{-4} - 2\sin 4}\right].$$

This implies that $u_1(t)$ is the solution to (3.33) (3.34) (3.35) (3.36) such that $u_1^{(3)}(0) = -8\,R + 16\,S$, and $u_2(t)$ is the solution to (3.33) (3.34) (3.35) (3.36) such that $u_2^{(3)}(0) = -27\,(B - C + D)$.

The starting high shot, $u_1(t)$ and the starting low shot, $u_2(t)$, are shown in fig. 3.1 with the first few shots.

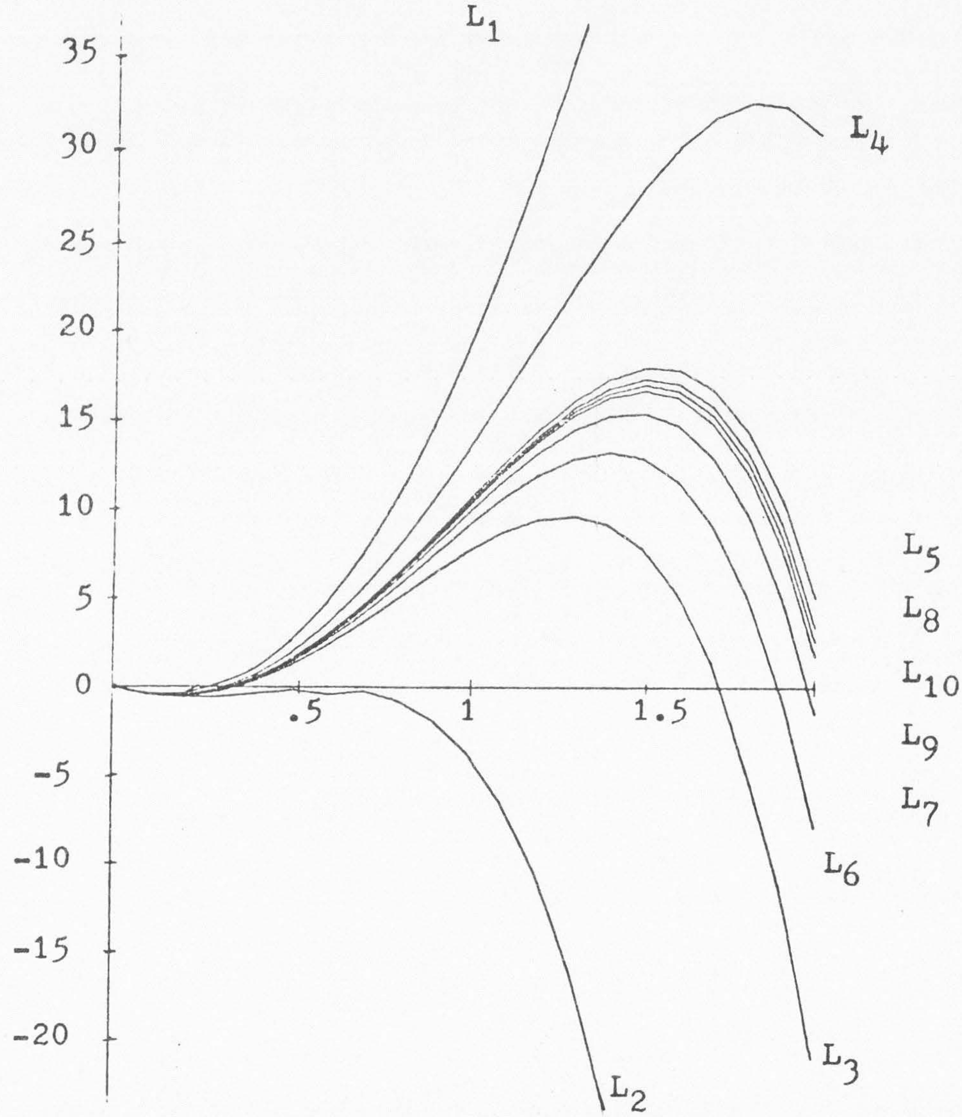

Figure 3.1. Starting high "shot", L_1 , starting low "shot" L_2, with the first eight "shots" L_3 - L_{10}.

Example 2

The problem

$$y^{(4)}(t) + \left[\frac{t+8}{2}\right] y^{(3)}(t) + 4t\, y^{(2)}(t) + (t^2 + 4t - 16)\, y^{(1)}(t)$$

$$+ \left[\frac{3t^2 + 2t - 48}{3}\right] y(t) = 0 \qquad (3.38)$$

$$y(0) = 36 \qquad (3.39)$$

$$y^{(1)}(0) = 10 \qquad (3.40)$$

$$y^{(3)}(0) = 12 \qquad (3.41)$$

$$y(2) = 0 \qquad (3.42)$$

will also be used to illustrate the shooting method with bisection. The same Lipschitz condition as described in example 1 will be used in this example. First, the Lipschitz constants will be given. Then the bounding functions, $v_1(t)$ and $v_2(t)$, will be found and finally, the iterative proceedure will be used.

The Lipschitz constants are obviously

$$\begin{array}{ll} L_0 = -16 & K_0 = 24 \\ L_1 = -16 & K_1 = 44 \\ L_2 = 0 & K_2 = 24 \\ L_3 = 4 & K_3 = 7 \,. \end{array}$$

The bounding curves, $v_1(t)$ and $v_2(t)$, are solutions to

$$v_2^{(4)}(t) + 7\, v_2^{(3)}(t) + 24\, v_2^{(2)}(t) + 44\, v_2^{(1)}(t) + 24\, v_2(t) = 0$$

and

$$v_1^{(4)}(t) + 4\, v_1^{(3)}(t) - 16\, v_1^{(1)}(t) - 16\, v_1(t) = 0$$

that satisfies the boundary conditions (3.39) (3.40) (3.41) (3.42). Actually,

$$v_1(t) = A\, e^{2t} + (B\, t^2 + C\, t + D)\, e^{-2t}$$

and

$$v_2(t) = R\, e^{-3t} + (S\, t^2 + Q\, t\, P)\, e^{-2t}$$

where

$$A = \left[\frac{1322\, e^{-4}}{59\, e^{-4} - 3\, e^{4}}\right]$$

$$B = \left[\frac{-207\, e^{-4} - 1065\, e^{4}}{354\, e^{-4} - 18\, e^{4}}\right]$$

$$C = \left[\frac{-332\, e^{-4} - 252\, e^{4}}{59\, e^{-4} - 3\, e^{4}}\right]$$

$$D = \left[\frac{802\, e^{-4} - 108\, e^{4}}{59\, e^{-4} - 3\, e^{4}}\right]$$

$$R = \left[\frac{-1284}{3\, e^{-2} - 4}\right]$$

$$S = \left[\frac{521 + 171\, e^{-2}}{3\, e^{-2} - 4}\right]$$

$$Q = \left[\frac{246\, e^{-2} - 1612}{3\, e^{-2} - 4}\right]$$

$$P = \left[\frac{1140 + 108\, e^{-2}}{3\, e^{-2} - 4}\right]$$

This implies that $u_1(t)$ is the solution to (3.38) (3.39) (3.40) (3.41) such that $u_1^{(2)}(0) = 4 A + 2 B - 4 C + 4 D$ and $u_2(t)$ is the solution to (3.38) (3.39) (3.40) (3.41) such that $u_2^{(2)}(0) = 9 A + 2 B - 4 C + 4 D$.

The starting high shot, $u_1(t)$, and the starting low shot, $u_2(t)$, are shown in fig. 3.2 with the first few shots.

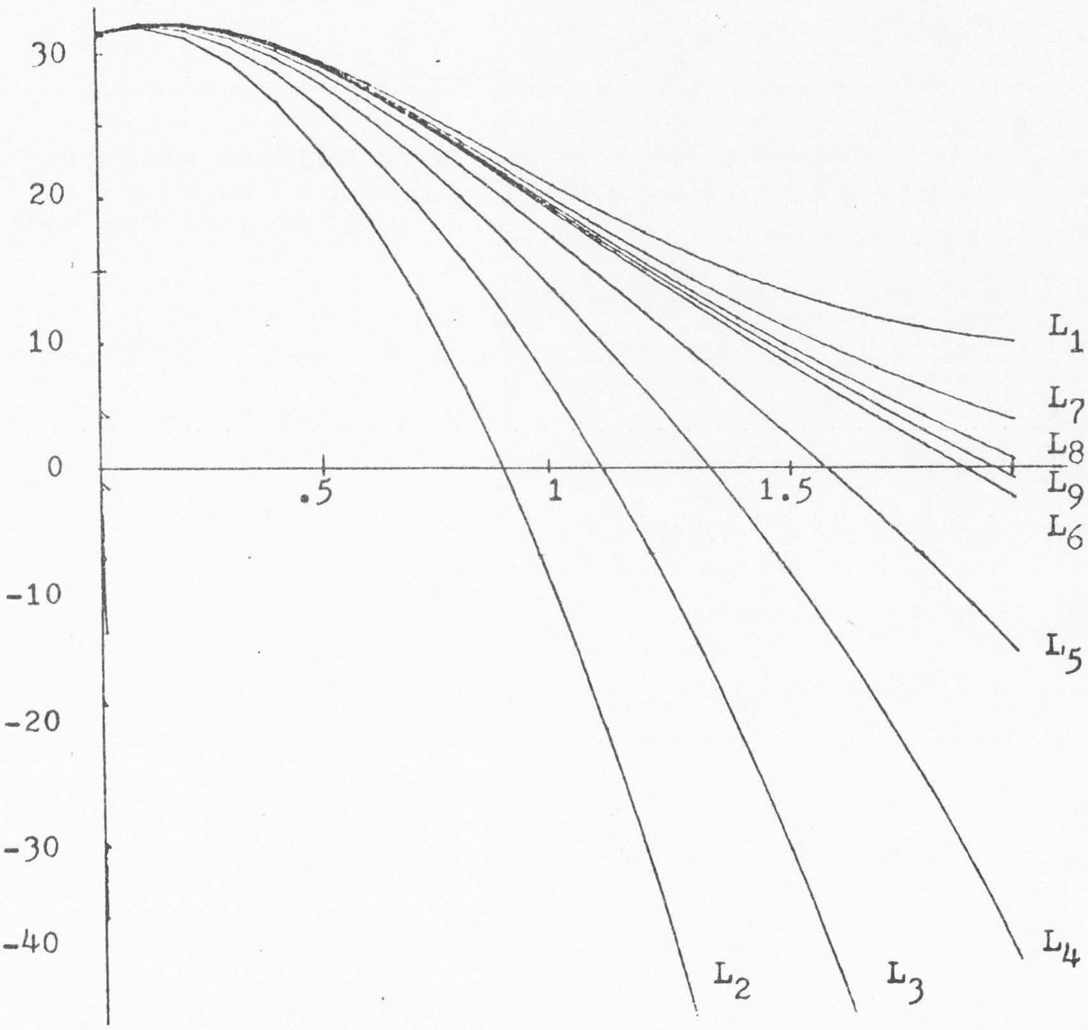

Figure 3.2. The starting high "shot" L_1, the starting low "shot" L_2, and the first seven "shots" L_3 - L_9.

CHAPTER IV

EVEN ORDER SHOOTING METHOD

4.1 Introduction

In this chapter it will be assumed that n is an even number and that $f(t,u,u^{(1)},\ldots,u^{(n-1)})$ is continuous on the region

$$a,b \ \ x\ R^{n}.$$

Furthermore, it shall be assumed that the differential equation

$$u^{(n)}(t) + f(t,u(t),u^{(1)}(t),\ldots,u^{(n-1)}(t)) = 0 \qquad (4.1)$$

has the properties that

(1) all initial value problems have unique solutions which exist over the interval $[a,b]$, and

(2) that no two solutions to the differential equation (4.1) can agree in value at more than one point in $[a,b]$, that is, the two-point boundary value problem with (4.1) has unique solutions, if they exist.

First a comparison theorem will be proven, then an uniqueness theorem for the two-point boundary value problem with (4.1) will be proven. Finally the shooting method will be developed for the two-point boundary value problem with (4.1).

4.2 Theorems of Comparison and Existence

Theorem 4.1 (Comparison of solutions)

Let $f(t,u,u^{(1)},\ldots,u^{(n-1)})$ be continuous in $(t,u,u^{(1)},\ldots,u^{(n-1)})$ and $v(t)$ be a solution to

$$v^{(n)}(t) + f(t,v(t),v^{(1)}(t),\ldots,v^{(n-1)}(t)) > 0. \qquad (4.2)$$

(1) If $u(t)$ is the solution to (4.1) that agrees with $v(t)$ in value and n-1 derivatives at some point $t_0 \in [a,b]$, then

$$v(t) > u(t) \qquad t \neq t_0. \qquad (4.3)$$

(2) If $u(t)$ is the solution to (4.1) that agrees with $v(t)$ in value at both a and b , then

$$u(t) \geqq v(t) \qquad t \neq a,b. \qquad (4.4)$$

The inequality signs may be reversed throughout this theorem.

Lemma 4.1 contains the basic information of the above theorem. It says, roughly, that statement (1) is true for t sufficiently near t_0.

Lemma 4.1

Let $v(t)$ be n times continuously differentiable function on $[a,b]$ which satisfies (4.2). For $(t,s) \in [a,b] \times [a,b]$, let $u(t,s)$ denote the solution of (4.1) which agrees with $v(t)$ in value and n-1 derivatives at $t = s$. Then for every $s_0 \in [a,b]$ there exist a $\delta_0 > 0$ such that whenever $(t,s) \in [a,b] \times [a,b]$ with

$$|t - s_0| < \delta_0 \qquad \text{and} \qquad |s - s_0| < \delta_0$$

then

$$u^{(n)}(t,s) < v^{(n)}(t) \tag{4.5}$$

if k is an odd number less than n

$$u^{(k)}(t,s) < v^{(k)}(t) \qquad t > s \tag{4.6}$$

$$u^{(k)}(t,s) > v^{(k)}(t) \qquad t < s, \tag{4.7}$$

if k is an even number less than n

$$u^{(k)}(t,s) < v^{(k)}(t) \qquad t \neq s \tag{4.8}$$

Proof. The proof of statement (4.5) follows the format of the proof of statement (3.5) in lemma 3.1. The only difference is that $u^{(4)}(t,s)$ is replaced by $u^{(n)}(t,s)$ and $v^{(4)}(t)$ is replaced by $v^{(n)}(t)$. (4.6) is derived from (4.5) if $k = n-1$ and from (4.8) otherwise by integrating from t to s. Similarly, (4.7) is derived by integrating from s to t. (4.8) is derived from (4.6) and (4.7) by integrating from s to t and t to s, respectively.

Proof of theorem 4.1. This proof is exactly the same as the proof of theorem 3.1 (see page 35) with the execption that lemma 4.1 must be used in place of lemma 3.1.

Theorem 4.2 (Existence and boundedness)

Let $u_1(t)$ and $u_2(t)$ be solutions to

$$u^{(n)}(t) + f(t,u(t),u^{(1)}(t),\dots,u^{(n-1)}(t)) = 0 \tag{4.1}$$

$$u(a) = A \tag{4.9}$$

then

$$u^{(n)}(t,s) < v^{(n)}(t) \qquad (4.5)$$

if k is an odd number less then n

$$u^{(k)}(t,s) < v^{(k)}(t) \qquad t > s \qquad (4.6)$$

$$u^{(k)}(t,s) > v^{(k)}(t) \qquad t < s \qquad (4.7)$$

if k is an even number less then n

$$u^{(k)}(t,s) < v^{(k)}(t) \qquad t \neq s. \qquad (4.8)$$

<u>Proof.</u> The proof of statement (4.5) follows the format of the proof of statement (3.5) in lemma 3.1. The only difference is that u (t,s) is replaced by u (t,s) and v (t) is replaced by v (t). (4.6) is derived from (4.5) if $k = n-1$ and from (4.8) otherwise by integrating from t to s. Similarly, (4.7) is derived by integrating from s to t. (4.8) is derived from (4.6) and (4.7) by integrating from s to t and t to s, respectively.

<u>Proof of theorem 4.1.</u> This proof is exactly the same as the proof of theorem 3.1 (see page 35) with the execption that lemma 4.1 must be used in place of lemma 3.1.

<u>Theorem 4.2</u> (Properties of existence and boundedness of solutions)

Let $u_1(t)$ and $u_2(t)$ be solutions to

$$u^{(n)}(t) + f(t,u(t),u^{(1)}(t), \dots ,u^{(n-1)}(t)) = 0 \qquad (4.1)$$

$$u(a) = A \qquad (4.9)$$

on $[a,b]$. Assume that $u_1(t) < u_2(t)$ for $t \neq a$.

(1) For every point $B \in (u_1(b), u_2(b))$, there exist a solution, $u(t)$, to (4.1) (4.9) such that

$$\text{a) } u(b) = B \tag{4.10}$$

$$\text{b) } (u^{(1)}(a), \ldots, u^{(n-1)}(a)) = c\,(u_1^{(1)}(a), \ldots, u_1^{(n-1)}(a)) + (1-c)\,(u_2^{(1)}(a), \ldots, u_2^{(n-1)}(a)) \tag{4.11}$$

where $c \in (0,1)$.

(2) If $u(t)$ is a solution to (4.1) (4.9) such that for $k = 1,2,\ldots,n-1$

$$u_1^{(k)}(a) \leq u^{(k)}(a) \leq u_2^{(k)}(a)$$

with strict inequality holding for at least one of the above conditions, then for $a < t \leq b$

$$u_1(t) < u(t) < u_2(t). \tag{4.12}$$

<u>Proof.</u> First (1) will be proven. By theorem 1.3, the solutions to the initial value problems with (4.1) are continuously dependent upon their initial conditions. By the intermediate value theorem, there exist a point

$$(m_1, \ldots, m_{n-1}) = c\,(u_1^{(1)}(a), \ldots, u_1^{(n-1)}(a)) + (1-c)\,(u_2^{(1)}(a), \ldots, u_2^{(n-1)}(a))$$

where c (0,1) such that the solution to the initial value problem (4.1) (4.9) with $(u^{(1)}(a), \ldots, u^{(n-1)}(a)) = (m_1, \ldots, m_{n-1})$ is the desired solution that satisfies (4.10) and (4.11).

Next to prove (2), by condradiction assume that there exist a $t_0 \in (a,b]$ such that

$$u_2(t_0) = u(t_0)$$

but this condradicts the assumption that no two solutions to (4.1) can meet more than once in $[a,b]$. By the continuity of the solutions of (4.1),

$$u(t) < u_2(t)$$

for $a < t \leq b$. A similar argument will show that

$$u_1(t) < u(t)$$

for $a < t \leq b$. Therefore, (4.12) holds and the proof of this theorem is finished.

4.3 Developing the Shooting Method

The shooting method for the n-th order differential equation two-point boundary value problem can be set up in the following manner. The function $f(t,u,u^{(1)},\ldots,u^{(n-1)})$ has been assumed to satisfy the Lipschitz condition (1.4) and f is defined for the point $(t,0,0,\ldots,0)$. This implies the equations

$$f(t,u,u^{(1)},\ldots,u^{(n-1)}) < f(t,0,\ldots,0) + K_0 \left|u\right| + \sum_{i=1}^{n-1} K_i \left|u^{(i)}\right|$$

and

$$f(t,u,u^{(1)},\ldots,u^{(n-1)}) > f(t,0,\ldots,0) - K_0 \left|u\right| + \sum_{i=1}^{n-1} K_i \left|u^{(i)}\right|$$

are true $(t,u,u^{(1)},\ldots,u^{(n-1)})$ in the domain of f. Adding $u^{(n)}$ to both sides of each inequality yields

$$u^{(n)} + f(t,u,u^{(1)},\ldots,u^{(n-1)}) < u^{(n)} + \sum_{i=1}^{n-1} K_i \left|u^{(i)}\right| + K_0 \left|u\right| + f(t,0,\ldots,0)$$

$$u^{(n)} + f(t,u,u^{(1)},\ldots,u^{(n-1)}) > u^{(n)} - \sum_{i=1}^{n-1} K_i \left|u^{(i)}\right| - K_0 \left|u\right| + f(t,0,\ldots,0).$$

Let $u(t)$, $v_1(t)$, $v_2(t)$ be the solutions to

$$u^{(n)}(t) + f(t,u(t),u^{(1)}(t),\ldots,u^{(n-1)}(t)) = 0$$

$$v_1^{(n)}(t) + \sum_{i=1}^{n-1} K_i\, v_1^{(i)}(t) + K_0\, v_1(t) + f(t,0,\ldots,0) = 0$$

and

$$v_2^{(n)}(t) - \sum_{i=1}^{n-1} K_i\, v_2^{(i)}(t) - K_0\, v_2(t) + f(t,0,\ldots,0) = 0,$$

respectively, that satisfies the specified boundary conditions of the two-point boundary value problem. Now

$$v_1^{(n)}(t) + f(t, v_1(t), v_1^{(1)}(t), \ldots, v_1^{(n-1)}(t)) > 0$$

and

$$v_2^{(n)}(t) + f(t, v_2(t), v_2^{(1)}(t), \ldots, v_2^{(n-1)}(t)) < 0$$

so by theorem 4.1 (2)

$$v_1(t) \leq v(t) \leq v_2(t) \qquad (4.13)$$

for $a < t < b$.

Let $u_1(t)$ and $u_2(t)$ be solutions to (4.1) that agree with $v_1(t)$ and $v_2(t)$, respectively, in value and n-1 derivatives at $t = a$. By theorem 4.1 (1) and (4.13)

$$u_1(t) < u(t) < u_2(t) \qquad a < t \leq b.$$

The proceedure now that $u_1(t)$ and $u_2(t)$ have been found is the following iterative scheme.

(1) Let $m_k = \left[u_1^{(k)}(a) + u_2^{(k)}(a)\right] / 2$ for $k = 1, 2, \ldots, n-1$.

(2) Find the solution, $v(t)$, to the initial value problem

$$v^{(n)}(t) + f(t, v(t), v^{(1)}(t), \ldots, v^{(n-1)}(t)) = 0 \qquad (4.1)$$

$$v^{(n-1)}(a) = m_{n-1} \qquad (4.14)$$

$$\vdots$$

$$v^{(1)}(a) = m_1$$

$$v(a) = A$$

Theorem 4.2 (2) says that

$$u_1(t) < v(t) < u_2(t)$$

for $a < t \leq b$.

(3) If $v(b) = B$ stop, for the desired solution has been found. If $v(b) \neq B$, then replace $u_1(t)$ by $v(t)$ if $v(b) < B$, otherwise replace $u_2(t)$ by $v(t)$ and repeat the above steps. The above process is terminated when $v(b)$ becomes sufficiently close to B.

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